TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 354, Number 3, Pages 1193–1213
S 0002-9947(01)02913-0
Article electronically published on November 1, 2001

SHARP SOBOLEV-POINCARÉ INEQUALITIES ON COMPACT RIEMANNIAN MANIFOLDS

EMMANUEL HEBEY

Abstract. Given (M, g) a smooth compact Riemannian n-manifold, $n \geq 3$, we return in this article to the study of the sharp Sobolev-Poincaré type inequality

$$\|u\|_{2^\star}^2 \leq K_n^2 \|\nabla u\|_2^2 + B\|u\|_1^2 \tag{0.1}$$

where $2^\star = 2n/(n-2)$ is the critical Sobolev exponent, and K_n is the sharp Euclidean Sobolev constant. Druet, Hebey and Vaugon proved that (0.1) is true if $n = 3$, that (0.1) is true if $n \geq 4$ and the sectional curvature of g is a nonpositive constant, or the Cartan-Hadamard conjecture in dimension n is true and the sectional curvature of g is nonpositive, but that (0.1) is false if $n \geq 4$ and the scalar curvature of g is positive somewhere. When (0.1) is true, we define $B(g)$ as the smallest B in (0.1). The saturated form of (0.1) reads as

$$\|u\|_{2^\star}^2 \leq K_n^2 \|\nabla u\|_2^2 + B(g)\|u\|_1^2. \tag{0.2}$$

We assume in this article that $n \geq 4$, and complete the study by Druet, Hebey and Vaugon of the sharp Sobolev-Poincaré inequality (0.1). We prove that (0.1) is true, and that (0.2) possesses extremal functions when the scalar curvature of g is negative. A fairly complete answer to the question of the validity of (0.1) under the assumption that the scalar curvature is not necessarily negative, but only nonpositive, is also given.

Let (M, g) be a smooth compact Riemannian n-dimensional manifold, $n \geq 3$, and $H_1^2(M)$ be the Sobolev space defined as the completion of $C^\infty(M)$ with respect to

$$\|u\|_{H_1^2}^2 = \int_M |\nabla u|^2 dv_g + \int_M u^2 dv_g.$$

Also let K_n be the best constant K in the Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star} dx\right)^{\frac{1}{2^\star}} \leq K \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\frac{1}{2}}$$

where $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent. It is well known that

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}$$

Received by the editors November 18, 2000.
2000 *Mathematics Subject Classification.* Primary 58E35.

where ω_n is the volume of the unit sphere S^n of $\mathbb{R}^{n+1}$. By the Sobolev-Poincaré inequality, there exists $A, B > 0$, such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq A \int_M |\nabla u|^2 dv_g + B \left(\int_M |u| dv_g\right)^2 . \tag{1.1}$$

Basic arguments as in the proof of proposition 4.5 of Hebey [10] lead to the following: any constant A in (1.1) is such that $A \geq K_n^2$, and for any $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq \left(K_n^2 + \varepsilon\right) \int_M |\nabla u|^2 dv_g + B_\varepsilon \left(\int_M |u| dv_g\right)^2 . \tag{1.2}$$

It follows that the best constant A in (1.1) is $A = K_n^2$. In this article we consider the sharp inequality

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \left(\int_M |u| dv_g\right)^2 \tag{1.3}$$

where $u \in H_1^2(M)$ and $B > 0$ is independent of u. We say that (1.3) is valid if there exists B such that (1.3) holds for all $u \in H_1^2(M)$. Concerning the terminology, a Cartan-Hadamard n-dimensional manifold is a complete simply-connected n-dimensional Riemannian manifold of nonpositive sectional curvature. We then refer to the n-dimensional Cartan-Hadamard conjecture as follows: given $(\tilde{M}, \tilde{g})$ a Cartan-Hadamard n-dimensional manifold, and Ω a smooth bounded domain in $\tilde{M}$, we ask for the inequality

$$\frac{Area_{\tilde{g}}(\partial\Omega)}{Vol_{\tilde{g}}(\Omega)^{(n-1)/n}} \geq n \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n}}$$

that is the value one gets for the above ratio in the Euclidean n-dimensional space when Ω is a ball. Such a conjecture is true for $n = 2$ thanks to Weil [14], for $n = 3$ thanks to Kleiner [12], and for $n = 4$ thanks to Croke [3]. The following assertions were proved by Druet, Hebey and Vaugon [7]: given (M, g) a compact Riemannian manifold of dimension $n \geq 3$,

1. If $n = 3$, then (1.3) is valid without any assumption on the manifold.
2. If $n \geq 4$ and the sectional curvature of g is a nonpositive constant, or the Cartan-Hadamard conjecture in dimension n is true and the sectional curvature of g is nonpositive, then (1.3) is valid.
3. If $n \geq 4$ and the scalar curvature of g is positive somewhere, then (1.3) is not valid.

In particular, since the Cartan-Hadamard conjecture is true in dimension 4, it follows from these assertions that inequality (1.3) when $n = 4$ is not valid if the scalar curvature of g is positive somewhere, but valid if the sectional curvature of g is nonpositive. Assume now that (1.3) is valid. Then there exists $B > 0$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \left(\int_M |u| dv_g\right)^2 .$$

We define $B(g)$ as the smallest B in this inequality. It is easily seen that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le K_n^2 \int_M |\nabla u|^2 dv_g + B(g) \left(\int_M |u| dv_g\right)^2 \tag{1.4}$$

and taking $u \equiv 1$, we see that $B(g) \ge V_g^{-\frac{n+2}{n}}$ where V_g is the volume of M with respect to g. We say that $u \in H_1^2(M)$, $u \not\equiv 0$, is an extremal function for (1.4) if u realizes the equality in (1.4). Natural questions we address in this article are whether or not (1.3) is valid when the scalar curvature of g is negative, and whether or not (1.4) possesses extremal functions in this situation. We already know by Druet, Hebey and Vaugon [7] that (1.3) is valid in dimension 3 without any sign assumption on the curvature. When $n \ge 4$, we answer the above two questions in the following theorem:

Theorem 1. *Let (M, g) be a compact Riemannian manifold of dimension $n \ge 4$ and negative scalar curvature. Then inequality* (1.3) *is valid. Moreover, inequality* (1.4) *possesses extremal functions.*

Concerning the validity of (1.3), and combining this theorem with what was proved by Druet, Hebey and Vaugon [7], it follows that (1.3) is true when $n = 3$ without any assumptions on the sign of the curvature, and that when $n \ge 4$, (1.3) is true if the scalar curvature of g is negative, but false if the scalar curvature of g is positive somewhere. Then, a natural question is whether or not (1.3) is valid if the scalar curvature of g is nonpositive. Following standard terminology, we say that g is conformally flat if conformal changes of g induce local isometries with the Euclidean space. When $n \ge 4$, a necessary condition is that $W_g \equiv 0$, where W_g is the Weyl curvature tensor of g. When g is not conformally flat, we refer to nonconformally flat points x as points where $W_g(x) \not\equiv 0$. A fairly complete answer to the question of whether or not (1.3) is valid if the scalar curvature of g is nonpositive is given by the following theorem:

Theorem 2. *Let (M, g) be a compact Riemannian n-manifold of nonpositive scalar curvature, $n \ge 3$. If g is conformally flat, then inequality* (1.3) *is valid. Conversely, assume $n \ge 6$ and g is not conformally flat. If g is scalar flat in an open neighbourhood of one nonconformally flat point, then* (1.3) *is not valid.*

In particular, it follows from the second assertion of the theorem that if $n \ge 6$, g is scalar flat, and not conformally flat, then (1.3) is not valid. Possible general references for more information on the study of sharp inequalities of Sobolev type are Druet-Hebey [6] and Hebey [10]. Sections 1, 2, and 3 of this article are devoted to the proof of Theorem 1. Section 4 is devoted to the proof of Theorem 2.

1. The PDE aspect of the problem

Let α be a sequence of positive real numbers converging to some $\alpha_0 \in (0, +\infty]$. We assume that for any α in the sequence,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n^2} \tag{1.1}$$

where I_α is the functional defined on $H_1^2(M)\backslash\{0\}$ by

$$I_\alpha(u) = \frac{\|\nabla u\|_2^2 + \alpha \|u\|_1^2}{\|u\|_{2^\star}^2}.$$

We claim that (1.1) implies the existence of a nonnegative minimizer $u_\alpha \in H_1^2(M)$ for

$$\lambda_\alpha = \inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u).$$

We proceed as in Druet, Hebey and Vaugon [7]. For $q < 2^\star$, let $\theta_q > 1$ be such that θ_q goes to 1 as q goes to $2^\star$. We fix $\alpha > 0$ and for $q < 2^\star$ we let

$$\lambda_q = \inf_{H_1^2(M)\backslash\{0\}} \frac{\|\nabla u\|_2^2 + \alpha\|u\|_{\theta_q}^2}{\|u\|_q^2}. \tag{1.2}$$

The embedding of $H_1^2(M)$ in $L^q(M)$ being compact, and since the above functional is homogeneous, there exists a nonnegative minimizer u_q for λ_q such that $\|u_q\|_q = 1$. Clearly, u_q is a weak solution of

$$\Delta_g u_q + \alpha \left(\int_M u_q^{\theta_q} dv_g\right)^{\frac{2}{\theta_q}-1} u_q^{\theta_q - 1} = \lambda_q u_q^{q-1} \tag{1.3}$$

where Δ_g stands for the Laplacian with respect to g. It is easily checked that, up to a subsequence, we may assume that for some $\lambda'_\alpha \le \lambda_\alpha$, the sequence (λ_q) goes to λ'_α as q goes to $2^\star$. Noting that (u_q) is bounded in $H_1^2(M)$, there exists $u_\alpha \in H_1^2(M)$ such that, up to a subsequence, (u_q) converges weakly to u_α in $H_1^2(M)$, strongly to u_α in $L^2(M)$, and almost everywhere. Moreover, one can assume that

$$u_q^{q-1} \rightharpoonup u_\alpha^{2^\star - 1} \quad \text{in} \quad L^{2^\sharp}(M)$$

where $2^\sharp = 2n/(n+2)$ is the conjugate exponent of $2^\star$. By (1.2), and since for any $\varepsilon > 0$ there exists B_ε such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \le \left(K_n^2 + \varepsilon\right)\|\nabla u\|_2^2 + B_\varepsilon \|u\|_1^2$$

one has that $u_\alpha \not\equiv 0$. This is by now standard. Let $\varepsilon_q = \theta_q - 1$. Clearly, $(u_q^{\varepsilon_q})$ is bounded in $L^p(M)$ for any $p > 1$. On such an assertion, just note that for $q \gg 1$,

$$\left(\int_M u_q^{p\varepsilon_q} dv_g\right)^{1/p} \le \left(\int_M u_q^2 dv_g\right)^{\varepsilon_q/2} V_g^{\frac{1}{p} - \frac{\varepsilon_q}{2}} \tag{1.4}$$

where V_g is the volume of M with respect to g. Since L^p-spaces are reflexive for $p > 1$, there exists $\Sigma_\alpha \in \bigcap_{p>1} L^p(M)$ such that for any $p > 1$, and up to a subsequence,

$$u_q^{\varepsilon_q} \rightharpoonup \Sigma_\alpha \quad \text{in} \quad L^p(M).$$

Passing to the limit as q goes to $2^\star$ in (1.4), one gets that for any $p > 1$,

$$\|\Sigma_\alpha\|_p \le V_g^{1/p}.$$

As an easy consequence, $\Sigma_\alpha \in L^\infty(M)$ and $0 \le \Sigma_\alpha \le 1$. Another easy claim is that $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ having the property that $|\varphi| \le C u_\alpha$ on M for some constant $C > 0$. By passing to the limit in (1.3), one gets that u_α is a weak solution of

$$\Delta_g u_\alpha + \alpha\left(\int_M u_\alpha dv_g\right)\Sigma_\alpha = \lambda'_\alpha u_\alpha^{2^\star - 1}. \tag{1.5}$$

Clearly, $\|u_\alpha\|_{2^\star} \le 1$. Mutiplying (1.5) by u_α and integrating over M gives

$$\frac{\|\nabla u_\alpha\|_2^2 + \alpha\|u_\alpha\|_1^2}{\|u_\alpha\|_{2^\star}^2} = \lambda'_\alpha \|u_\alpha\|_{2^\star}^{2^\star - 2}.$$

This implies that $\|u_\alpha\|_{2^\star} = 1$ and that $\lambda'_\alpha = \lambda_\alpha$. In particular, u_α is a minimizer for λ_α, and the above claim is proved. Summarizing, let α be a sequence of positive real numbers converging to some $\alpha_0 \in (0, +\infty]$, and assume that for any α in the sequence,

$$\lambda_\alpha = \inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n^2}$$

where I_α is as above. Then there exists $u_\alpha \in H_1^2(M)$, $u_\alpha \geq 0$ and $u_\alpha \not\equiv 0$, such that

$$\Delta_g u_\alpha + \alpha \left(\int_M u_\alpha dv_g \right) \Sigma_\alpha = \lambda_\alpha u_\alpha^{2^\star - 1}$$

and $\int_M u_\alpha^{2^\star} dv_g = 1$, where $\Sigma_\alpha \in L^\infty(M)$ is such that $0 \leq \Sigma_\alpha \leq 1$ and $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ such that $|\varphi| \leq C u_\alpha$ for some constant $C > 0$.

2. Blow up and PDE estimates

Let α be a sequence of positive real numbers converging to some $\alpha_0 \in (0, +\infty]$. We assume that for any α in the sequence,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n^2}$$

where I_α is as in Section 1. Thanks to Section 1, we get that there exist $u_\alpha \in H_1^2(M)$ and $\Sigma_\alpha \in L^\infty(M)$, $0 \leq \Sigma_\alpha \leq 1$, such that

$$\Delta_g u_\alpha + \alpha \left(\int_M u_\alpha dv_g \right) \Sigma_\alpha = \lambda_\alpha u_\alpha^{2^\star - 1} \tag{2.1}$$

and $\int_M u_\alpha^{2^\star} dv_g = 1$. Moreover, $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ such that $|\varphi| \leq C u_\alpha$ for some constant $C > 0$, and we have that

$$\alpha \int_M u_\alpha dv_g \int_M \Sigma_\alpha dv_g = \lambda_\alpha \int_M u_\alpha^{2^\star - 1} dv_g. \tag{2.2}$$

Let $u \in H_1^2(M)$, $u \geq 0$, be such that for any nonnegative $\varphi \in H_1^2(M)$,

$$\int_M (\nabla u \nabla \varphi)\, dv_g \leq \int_M u^{2^\star - 1} \varphi dv_g$$

where $(\nabla u \nabla \varphi)$ is the pointwise scalar product with respect to g of ∇u and $\nabla \varphi$. We know from PDE theory and the Moser iterative scheme that $u \in L^\infty(M)$, with the additional property that for any x in M, any $\Lambda > 0$, any $p > 0$, and any $q > 2^\star$, there exists $\delta > 0$ such that if

$$\int_{B_x(2\delta)} u^q dv_g \leq \Lambda,$$

then

$$\sup_{y \in B_x(\delta)} u(y) \leq \tilde{C} \left(\int_{B_x(2\delta)} u^p dv_g \right)^{1/p}$$

where $\tilde{C} > 0$ does not depend on u. It follows that $u_\alpha \in L^\infty(M)$. In particular, $u_\alpha \in H_2^p(M)$ for any $p > 1$, and it follows from (2.1) that u_α is actually in $C^{1,\lambda}$ for any $\lambda \in (0,1)$. As another remark, the sequence (u_α) is bounded in $H_1^2(M)$. We assume in what follows that

$$\lim_{\alpha\to\alpha_0}\int_M u_\alpha^2 dv_g = 0. \tag{2.3}$$

This is automatically the case if $\alpha_0 = +\infty$. Multiplying (2.1) by u_α, and integrating over M, we get indeed that

$$\|\nabla u_\alpha\|_2^2 + \alpha\|u_\alpha\|_1^2 = \lambda_\alpha.$$

As a consequence, $\|u_\alpha\|_1 \to 0$ as $\alpha \to +\infty$, and by Hölder's inequality, since u_α is of norm 1 in $L^{2^\star}$, this implies that $\|u_\alpha\|_2 \to 0$ as $\alpha \to +\infty$. By Hebey-Vaugon [11], there exists $B \in \mathbb{R}$ such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star} \le K_n^2\int_M |\nabla u|^2 dv_g + B\int_M u^2 dv_g.$$

Taking $u = u_\alpha$ in this inequality, we get that

$$1 \le \lambda_\alpha K_n^2 + B\int_M u_\alpha^2 dv_g$$

and it follows from this inequality and (2.3) that

$$\lim_{\alpha\to\alpha_0}\lambda_\alpha = \frac{1}{K_n^2}. \tag{2.4}$$

Similarly,

$$1 - B\int_M u_\alpha^2 dv_g \le K_n^2\int_M |\nabla u_\alpha|^2 dv_g = K_n^2\lambda_\alpha - K_n^2\alpha\left(\int_M u_\alpha dv_g\right)^2$$

and it follows that

$$\lim_{\alpha\to\alpha_0}\alpha\left(\int_M u_\alpha dv_g\right)^2 = 0. \tag{2.5}$$

Define now a concentration point x for (u_α) by the property that for any $\delta > 0$,

$$\limsup_{\alpha\to\alpha_0}\int_{B_x(\delta)} u_\alpha^{2^\star} dv_g > 0.$$

As in Druet, Hebey and Vaugon [7], (u_α) has, up to a subsequence, one and only one concentration point x_0. One may then assume that for any $\delta > 0$,

$$\lim_{\alpha\to\alpha_0}\int_{B_{x_0}(\delta)} u_\alpha^{2^\star} dv_g = 1$$

and we have that

$$u_\alpha \to 0 \quad \text{in } C^0_{loc}(M\backslash\{x_0\}) \tag{2.6}$$

as α goes to α_0. We let $x_\alpha \in M$ and $\mu_\alpha \in \mathbb{R}$ be such that

$$u_\alpha(x_\alpha) = \|u_\alpha\|_\infty = \mu_\alpha^{-\frac{n-2}{2}}.$$

According to what we just said, $x_\alpha \to x_0$ and $\mu_\alpha \to 0$ as $\alpha \to \alpha_0$. By (2.5), noting that

$$1 = \|u_\alpha\|_{2^\star}^{2^\star} \le \|u_\alpha\|_\infty^{2^\star-1}\|u_\alpha\|_1$$

we get that

$$\lim_{\alpha\to\alpha_0} \alpha\mu_\alpha^{\frac{n+2}{2}} \int_M u_\alpha dv_g = 0. \tag{2.7}$$

Taking inspiration from different works developed by Druet (see Druet [5], Druet and Robert [8], and Druet and Hebey [6] for an overview), we prove in this section several estimates for the u_α's. Theorem 1 follows from these estimates as explained in the next section. As a first estimate, we claim that the following holds:

ESTIMATE 1. For any $R > 0$,

$$\lim_{\alpha\to\alpha_0} \int_{B_{x_\alpha}(R\mu_\alpha)} u_\alpha^{2^\star} dv_g = 1 - \varepsilon_R \tag{2.8}$$

where $\varepsilon_R > 0$ is such that $\varepsilon_R \to 0$ as $R \to +\infty$.

Proof of Estimate 1. We let $\exp_{x_\alpha}$ be the exponential map at x_α. There clearly exists $\delta > 0$, independent of α, such that for any α, $\exp_{x_\alpha}$ is a diffeomorphism from $B_0(\delta) \subset \mathbb{R}^n$ onto $B_{x_\alpha}(\delta)$. For $x \in B_0(\mu_\alpha^{-1}\delta)$, we set

$$\tilde{g}_\alpha(x) = \left(\exp_{x_\alpha}^\star g\right)(\mu_\alpha x), \quad \tilde{u}_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}} u_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)$$

and $\tilde{\Sigma}_\alpha(x) = \Sigma_\alpha\left(\exp_{x_\alpha}(\mu_\alpha x)\right)$. It is easily seen that

$$\Delta_{\tilde{g}_\alpha}\tilde{u}_\alpha + \alpha\mu_\alpha^{\frac{n+2}{2}}\left(\int_M u_\alpha dv_g\right)\tilde{\Sigma}_\alpha = \lambda_\alpha \tilde{u}_\alpha^{2^\star-1}. \tag{2.9}$$

Moreover,

$$\tilde{u}_\alpha(0) = \|\tilde{u}_\alpha\|_\infty = 1 \tag{2.10}$$

and if ξ stands for the Euclidean metric of $\mathbb{R}^n$,

$$\lim_{\alpha\to\alpha_0} \tilde{g}_\alpha = \xi \quad \text{in } C^2(K) \tag{2.11}$$

for any compact subset K of $\mathbb{R}^n$. Thanks to (2.7) and (2.9)-(2.11), we get by standard elliptic theory, as developed in Gilbarg-Trudinger [9], that there exists some $\tilde{u} \in C^1(\mathbb{R}^n)$ such that for any compact subset K of $\mathbb{R}^n$,

$$\lim_{\alpha\to\alpha_0} \tilde{u}_\alpha = \tilde{u} \quad \text{in } C^1(K). \tag{2.12}$$

Clearly, $\tilde{u}(0) = 1$ and $\tilde{u} \not\equiv 0$. Moreover, it is easily seen that $\tilde{u} \in H_{0,1}^2(\mathbb{R}^n)$, where $H_{0,1}^2(\mathbb{R}^n)$ is the homogeneous Euclidean Sobolev space of order two for integration and order one for differentiation. By passing to the limit as α goes to α_0 in (2.9), according to (2.4), (2.7), (2.11), and (2.12), we get that $\tilde{u}$ is a solution of

$$\Delta_\xi \tilde{u} = \frac{1}{K_n^2}\tilde{u}^{2^\star-1}.$$

By Caffarelli-Gidas-Spruck [2], and also Obata [13],

$$\tilde{u}(x) = \left(\frac{1}{1+A|x|^2}\right)^{\frac{n-2}{2}} \tag{2.13}$$

where $A^{-1} = n(n-2)K_n^2$, since $\tilde{u}(0) = \|\tilde{u}\|_\infty = 1$ by (2.10) and (2.12). Noting that $\tilde{u}$ is of norm 1 in $L^{2^\star}(\mathbb{R}^n)$, and that for any $R > 0$,

$$\int_{B_{x_\alpha}(R\mu_\alpha)} u_\alpha^{2^\star} dv_g = \int_{B_0(R)} \tilde{u}_\alpha^{2^\star} dv_{\tilde{g}_\alpha},$$

we get that

$$\lim_{\alpha\to\alpha_0}\int_{B_{x_\alpha}(R\mu_\alpha)} u_\alpha^{2^\star}\,dv_g = 1-\int_{\mathbb{R}^n\setminus B_0(R)}\tilde{u}^{2^\star}\,dx.$$

This proves (2.8). □

We now claim that the following estimate holds:

ESTIMATE 2. There exists $C>0$, such that for any α, and any x,

$$d_g(x_\alpha,x)^{\frac{n}{2}-1}u_\alpha(x)\le C \tag{2.14}$$

where d_g is the distance with respect to g.

Proof of Estimate 2. We set

$$v_\alpha(x)=d_g(x_\alpha,x)^{\frac{n}{2}-1}u_\alpha(x)$$

and assume by contradiction that for some subsequence,

$$\lim_{\alpha\to\alpha_0}\|v_\alpha\|_\infty=+\infty. \tag{2.15}$$

Let y_α be some point in M where v_α is maximum. By (2.6), $y_\alpha\to x_0$ as $\alpha\to\alpha_0$, while by (2.15),

$$\lim_{\alpha\to\alpha_0}\frac{d_g(x_\alpha,y_\alpha)}{\mu_\alpha}=+\infty. \tag{2.16}$$

Fix now $\delta>0$ small, and set

$$\Omega_\alpha=u_\alpha(y_\alpha)^{\frac{2}{n-2}}\exp_{y_\alpha}^{-1}\left(B_{x_\alpha}(\delta)\right).$$

For $x\in\Omega_\alpha$, define

$$\tilde{v}_\alpha(x)=u_\alpha(y_\alpha)^{-1}u_\alpha\left(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x)\right)$$

and

$$h_\alpha(x)=\left(\exp_{y_\alpha}^\star g\right)\left(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x\right).$$

It easily follows from (2.15), since M is compact, that $u_\alpha(y_\alpha)\to+\infty$ as $\alpha\to\alpha_0$. Hence,

$$\lim_{\alpha\to\alpha_0}h_\alpha=\xi\quad\text{in } C^2\left(B_0(2)\right) \tag{2.17}$$

where ξ is the Euclidean metric. Independently, we have that

$$\Delta_{h_\alpha}\tilde{v}_\alpha\le\lambda_\alpha\tilde{v}_\alpha^{2^\star-1}. \tag{2.18}$$

Since $v_\alpha(y_\alpha)$ goes to $+\infty$, for α close to α_0, and all $x\in B_0(2)$,

$$d_g\left(x_\alpha,\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x)\right)\ge\frac{1}{2}d_g(x_\alpha,y_\alpha). \tag{2.19}$$

This implies that

$$\begin{aligned}\tilde{v}_\alpha(x)&\le 2^{\frac{n}{2}-1}d_g(x_\alpha,y_\alpha)^{1-\frac{n}{2}}u_\alpha(y_\alpha)^{-1}v_\alpha\left(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}}x)\right)\\&\le 2^{\frac{n}{2}-1}d_g(x_\alpha,y_\alpha)^{1-\frac{n}{2}}u_\alpha(y_\alpha)^{-1}v_\alpha(y_\alpha)\end{aligned}$$

so that for α close to α_0,

$$\sup_{x\in B_0(2)}\tilde{v}_\alpha(x)\le 2^{\frac{n}{2}-1}. \tag{2.20}$$

By (2.16) and (2.19), given $R > 0$, and for α close to α_0,

$$B_{y_\alpha}(2u_\alpha(y_\alpha)^{-\frac{2}{n-2}}) \cap B_{x_\alpha}(R\mu_\alpha) = \emptyset. \tag{2.21}$$

Noting that

$$\int_{B_0(2)} \tilde{v}_\alpha^{2^\star}\, dv_{h_\alpha} = \int_{B_{y_\alpha}(2u_\alpha(y_\alpha)^{-\frac{2}{n-2}})} u_\alpha^{2^\star}\, dv_g$$

it follows from (2.8) and (2.21) that

$$\lim_{\alpha\to\alpha_0} \int_{B_0(2)} \tilde{v}_\alpha^{2^\star}\, dv_{h_\alpha} = 0. \tag{2.22}$$

By (2.17), (2.18), (2.20), (2.22), and the Moser iterative scheme we get that

$$\lim_{\alpha\to\alpha_0} \sup_{x\in B_0(1)} \tilde{v}_\alpha(x) = 0.$$

But $\tilde{v}_\alpha(0) = 1$, so that (2.15) must be false. This proves (2.14). $\square$

Another estimate we need, somehow similar to the preceding one, is the following estimate:

Estimate 3. Given $R > 0$,

$$\sup_{x\in M\setminus B_{x_\alpha}(R\mu_\alpha)} d_g(x_\alpha, x)^{\frac{n}{2}-1} u_\alpha(x) = \varepsilon_R(\alpha) \tag{2.23}$$

where d_g is the distance with respect to g, and $\lim_{R\to+\infty} \lim_{\alpha\to\alpha_0} \varepsilon_R(\alpha) = 0$.

Proof of Estimate 3. We set

$$v_\alpha(x) = d_g(x_\alpha, x)^{\frac{n}{2}-1} u_\alpha(x)$$

and proceed once more by contradiction. Then there exists $y_\alpha \in M$ and $\varepsilon_0 > 0$ such that

$$\lim_{\alpha\to\alpha_0} \frac{d_g(x_\alpha, y_\alpha)}{\mu_\alpha} = +\infty \text{ and } v_\alpha(y_\alpha) \geq \varepsilon_0.$$

As above, we fix $\delta > 0$ small, and set

$$\Omega_\alpha = u_\alpha(y_\alpha)^{\frac{2}{n-2}} \exp_{y_\alpha}^{-1}\left(B_{x_\alpha}(\delta)\right).$$

For $x \in \Omega_\alpha$, we define

$$\tilde{v}_\alpha(x) = u_\alpha(y_\alpha)^{-1} u_\alpha\left(\exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}} x)\right)$$

and

$$h_\alpha(x) = \left(\exp_{y_\alpha}^\star g\right)\left(u_\alpha(y_\alpha)^{-\frac{2}{n-2}} x\right).$$

Once again $\Delta_{h_\alpha}\tilde{v}_\alpha \leq \lambda_\alpha \tilde{v}_\alpha^{2^\star-1}$. As when proving (2.14), for any $x \in B_0(\frac{1}{2}\varepsilon_0^{\frac{2}{n-2}})$,

$$d_g(x_\alpha, z_\alpha) \geq \frac{1}{2} d_g(x_\alpha, y_\alpha)$$

and

$$\tilde{v}_\alpha(x) = u_\alpha(y_\alpha)^{-1} v_\alpha(z_\alpha) d_g(x_\alpha, z_\alpha)^{1-\frac{n}{2}}$$

where $z_\alpha = \exp_{y_\alpha}(u_\alpha(y_\alpha)^{-\frac{2}{n-2}} x)$. It follows from (2.14) that

$$\tilde{v}_\alpha(x) \leq C 2^{\frac{n}{2}-1} \varepsilon_0^{-1}.$$

Noting that for $R > 0$, and for α close to α_0,

$$B_{y_\alpha}\left(\frac{1}{2}\varepsilon_0^{\frac{2}{n-2}}u_\alpha(y_\alpha)^{-\frac{2}{n-2}}\right)\cap B_{x_\alpha}(R\mu_\alpha)=\emptyset$$

we conclude as when proving (2.14) that (2.23) holds. □

Now, we claim that the following estimate holds:
ESTIMATE 4. If $\alpha_0 = +\infty$, then, for any $\delta > 0$,

$$\lim_{\alpha\to\alpha_0}\frac{\int_{M\backslash B_{x_0}(\delta)}u_\alpha^2dv_g}{\int_M u_\alpha^2dv_g}=0. \tag{2.24}$$

In other words, L^2-concentration holds for u_α in any dimension when $\alpha_0 = +\infty$.

Proof of Estimate 4. We let $0 \le \eta \le 1$ be a smooth increasing radially symmetric function with respect to x_0 such that $\eta = 1$ in $M\backslash B_{x_0}(\delta)$ and $\eta = 0$ in $B_{x_0}(\delta/2)$. By the Moser iterative scheme, see the beginning of this section, and by (2.6),

$$\int_{M\backslash B_{x_0}(\delta)}u_\alpha^2dv_g\le C_\delta\int_M u_\alpha dv_g\int_M\eta u_\alpha dv_g \tag{2.25}$$

where $C_\delta > 0$ is independent of α. Independently, we have by (2.1) that

$$\int_M u_\alpha dv_g\int_M\eta u_\alpha dv_g=\frac{1}{\alpha}\int_M\eta u_\alpha\left(\lambda_\alpha u_\alpha^{2^\star-1}-\Delta_g u_\alpha\right)dv_g. \tag{2.26}$$

Integrating by parts,

$$\int_M\eta u_\alpha\Delta_g u_\alpha dv_g=\int_M\eta|\nabla u_\alpha|^2dv_g+\int_M(\nabla\eta\nabla u_\alpha)\,u_\alpha dv_g$$

where $(\nabla\eta\nabla u_\alpha)$ is the pointwise scalar product with respect to g of $\nabla\eta$ and ∇u_α. Since $|(\nabla\eta\nabla u_\alpha)| \le |\nabla\eta||\nabla u_\alpha|$, and by Hölder's inequalities,

$$\int_M|(\nabla\eta\nabla u_\alpha)|\,u_\alpha dv_g\le\sqrt{\int_M u_\alpha^2dv_g}\sqrt{\int_{B_{x_0}(\delta)\backslash B_{x_0}(\delta/2)}|\nabla u_\alpha|^2dv_g}.$$

Coming back to (2.25) and (2.26), we get that

$$\begin{aligned}\frac{1}{C_\delta}\int_{M\backslash B_{x_0}(\delta)}u_\alpha^2dv_g&\le\frac{\lambda_\alpha}{\alpha}\int_M\eta u_\alpha^{2^\star}dv_g\\&+\frac{1}{\alpha}\sqrt{\int_M u_\alpha^2dv_g}\sqrt{\int_{M\backslash B_{x_0}(\delta/2)}|\nabla u_\alpha|^2dv_g}.\end{aligned}$$

Noting that

$$\int_M\eta u_\alpha^{2^\star}dv_g=\int_M(\eta u_\alpha^{2^\star-2})u_\alpha^2dv_g$$

it follows from (2.6) that

$$\lim_{\alpha\to+\infty}\frac{\int_M\eta u_\alpha^{2^\star}dv_g}{\int_M u_\alpha^2dv_g}=0.$$

Then, the proof of (2.24) reduces to the proof that for $\hat\delta > 0$ small, there exists $C > 0$, independent of α, such that

$$\int_{M\backslash B_{x_0}(\hat\delta)}|\nabla u_\alpha|^2dv_g\le C\int_M u_\alpha^2dv_g. \tag{2.27}$$

As above, let $0 \le \eta \le 1$ be a smooth function such that $\eta = 1$ in $M \backslash B_{x_0}(\hat{\delta})$ and $\eta = 0$ in $B_{x_0}(\hat{\delta}/2)$. Multiplying (2.1) by $\eta^2 u_\alpha$ and integrating over M, we get that

$$\int_M \eta^2 |\nabla u_\alpha|^2 dv_g + 2\int_M \eta u_\alpha \left(\nabla\eta\nabla u_\alpha\right) dv_g \le \lambda_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g .$$

Therefore,

$$\begin{aligned}\int_M \eta^2|\nabla u_\alpha|^2 dv_g &\le C\int_M |\eta\nabla u_\alpha| u_\alpha dv_g + \lambda_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g \\ &\le C\sqrt{\int_M u_\alpha^2 dv_g}\sqrt{\int_M \eta^2|\nabla u_\alpha|^2 dv_g} + \lambda_\alpha\int_M \eta^2 u_\alpha^{2^\star} dv_g\end{aligned}$$

and we get that

$$\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\int_M u_\alpha^2 dv_g} \le C\sqrt{\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\int_M u_\alpha^2 dv_g}} + \lambda_\alpha \frac{\int_M \eta^2 u_\alpha^{2^\star} dv_g}{\int_M u_\alpha^2 dv_g}.$$

Here again, by (2.6),

$$\lim_{\alpha\to+\infty} \frac{\int_M \eta u_\alpha^{2^\star} dv_g}{\int_M u_\alpha^2 dv_g} = 0$$

so that

$$\limsup_{\alpha\to+\infty} \frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\int_M u_\alpha^2 dv_g} \le C^2.$$

In particular, (2.27) holds, and this completes the proof of (2.24). □

At last, we claim that the following estimate holds:
Estimate 5. If $\alpha_0 < +\infty$ and $n \ge 4$, then, for any $\delta > 0$,

$$\lim_{\alpha\to\alpha_0} \frac{\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g}{\int_M u_\alpha^2 dv_g} = 0. \tag{2.28}$$

In other words, L^2-concentration holds for u_α in dimension $n \ge 4$ when $\alpha_0 < +\infty$.

Proof of Estimate 5. We clearly have that

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g = \int_{M\backslash B_{x_0}(\delta)} \Sigma_\alpha u_\alpha^2 dv_g .$$

Then, by the Moser iterative scheme (see the beginning of this section) and by (2.6),

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le C\int_M \Sigma_\alpha dv_g \left(\int_M u_\alpha dv_g\right)^2$$

where $C > 0$ is independent of α. By (2.2), and (2.4), we then get that

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le C\int_M u_\alpha dv_g \int_M u_\alpha^{2^\star-1} dv_g$$

and hence that

$$\int_{M\backslash B_{x_0}(\delta)} u_\alpha^2 dv_g \le C\sqrt{\int_M u_\alpha^2 dv_g}\int_M u_\alpha^{2^\star-1} dv_g \tag{2.29}$$

where $C > 0$ is independent of α. First, we suppose that $n \geq 6$. Then $2^\star - 1 \leq 2$, and we get by Hölder's inequalities that

$$\int_M u_\alpha^{2^\star - 1} dv_g \leq V_g^{\frac{3-2^\star}{2}} \left(\int_M u_\alpha^2 dv_g \right)^{\frac{2^\star - 1}{2}}$$

where V_g is the volume of M with respect to g. Coming back to (2.29) gives

$$\int_{M \setminus \mathcal{B}_\delta} u_\alpha^2 dv_g \leq C \left(\int_M u_\alpha^2 dv_g \right)^{\frac{2^\star}{2}} .$$

Since $2^\star > 2$, and $\|u_\alpha\|_2 \to 0$, we get that (2.28) holds when $n \geq 6$. Now we suppose that $n = 5$. Then $2 \leq 2^\star - 1 \leq 2^\star$, and we get by Hölder's inequalities that

$$\left(\int_M u_\alpha^{2^\star - 1} dv_g \right)^{\frac{1}{2^\star - 1}} \leq \left(\int_M u_\alpha^2 dv_g \right)^{\frac{s}{2}} \left(\int_M u_\alpha^{2^\star} dv_g \right)^{\frac{1-s}{2^\star}}$$

where

$$s = \frac{\frac{1}{2^\star - 1} - \frac{1}{2^\star}}{\frac{1}{2} - \frac{1}{2^\star}} = \frac{3}{2(2^\star - 1)} .$$

Coming back to (2.29), and since $\|u_\alpha\|_{2^\star} = 1$, we get that

$$\int_{M \setminus \mathcal{B}_\delta} u_\alpha^2 dv_g \leq C \left(\int_M u_\alpha^2 dv_g \right)^{\frac{5}{4}} .$$

Here again, $\|u_\alpha\|_2 \to 0$. This proves (2.28) when $n = 5$. At last, we suppose that $n = 4$. Then, $2^\star = 4$. We have that

$$\begin{aligned} \frac{\int_M u_\alpha^3 dv_g}{\sqrt{\int_M u_\alpha^2 dv_g}} &\leq \|u_\alpha\|_{L^\infty(M \setminus B_{x_\alpha}(\delta))} \sqrt{\int_M u_\alpha^2 dv_g} + \frac{\int_{B_{x_\alpha}(\delta)} u_\alpha^3 dv_g}{\sqrt{\int_{B_{x_\alpha}(\delta)} u_\alpha^2 dv_g}} \\ &\leq \varepsilon_\alpha + \frac{\int_{B_0(\delta\mu_\alpha^{-1})} \tilde{u}_\alpha^3 dv_{\tilde{g}_\alpha}}{\sqrt{\int_{B_0(\delta\mu_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha}}} \end{aligned}$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to \alpha_0$. For any $R > 0$, we get by the Cauchy-Schwarz inequality and (2.8) that

$$\int_{B_0(\delta\mu_\alpha^{-1})} \tilde{u}_\alpha^3 dv_{\tilde{g}_\alpha} \leq \int_{B_0(R)} \tilde{u}_\alpha^3 dv_{\tilde{g}_\alpha} + \varepsilon_R \sqrt{\int_{B_0(\delta\mu_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha}}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. It follows from these equations and (2.11) that for any $R > 0$,

$$\limsup_{\alpha \to \alpha_0} \frac{\int_M u_\alpha^3 dv_g}{\sqrt{\int_M u_\alpha^2 dv_g}} \leq \varepsilon_R + \frac{\int_{\mathbb{R}^n} \tilde{u}^3 dx}{\sqrt{\int_{B_0(R)} \tilde{u}^2 dx}}$$

where $\tilde{u}$ is as in (2.13). Noting that

$$\lim_{R \to +\infty} \int_{B_0(R)} \tilde{u}^2 dx = +\infty$$

when $n = 4$, this proves (2.28) when $n = 4$. □

3. Proof of Theorem 1

Suppose first that (1.3) is not valid. Let $\alpha_0 = +\infty$. Then, for any $\alpha \in (0, \alpha_0)$,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n^2}$$

where I_α is as in Section 1. Suppose now that (1.3) is valid and let $\alpha_0 = B(g)K_n^{-2}$. By the definition of $B(g)$, for any $\alpha \in (0, \alpha_0)$,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n^2}.$$

By Section 1 we get in both cases that there exist $u_\alpha \in H_1^2(M)$ and $\Sigma_\alpha \in L^\infty(M)$, $0 \le \Sigma_\alpha \le 1$, such that for a sequence (α), $\alpha < \alpha_0$, converging to α_0,

$$\Delta_g u_\alpha + \alpha \left(\int_M u_\alpha dv_g\right) \Sigma_\alpha = \lambda_\alpha u_\alpha^{2^\star - 1} \tag{3.1}$$

and

$$\int_M u_\alpha^{2^\star} dv_g = 1 \tag{3.2}$$

where $\lambda_\alpha < K_n^{-2}$ is the infimum of I_α. Moreover, $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ such that $|\varphi| \le C u_\alpha$ for some constant $C > 0$. As already mentioned in Section 2, if $\alpha_0 = +\infty$, we necessarily have that

$$\lim_{\alpha \to \alpha_0} \int_M u_\alpha^2 dv_g = 0. \tag{3.3}$$

On the other hand, let us assume that $\alpha_0 = B(g)K_n^{-2}$ and that (3.3) does not hold. Then, up to a subsequence, $u_\alpha \rightharpoonup u$ in $H_1^2(M)$ and $u_\alpha \to u$ in $L^2(M)$ as $\alpha \to \alpha_0$, where $u \in H_1^2(M)$, $u \not\equiv 0$. Up to another subsequence, we may also assume that $\lambda_\alpha \to \lambda$ as $\alpha \to \alpha_0$. We claim that u is an extremal function for (1.4). Indeed, since $0 \le \Sigma_\alpha \le 1$,

$$\lim_{\alpha \to \alpha_0} \int_M \Sigma_\alpha (u_\alpha - u)\, dv_g = 0$$

and we also have that $\int_M \Sigma_\alpha u_\alpha dv_g = \int_M u_\alpha dv_g$ and $\int_M u_\alpha dv_g \to \int_M u dv_g$ as $\alpha \to \alpha_0$. It follows that

$$\lim_{\alpha \to \alpha_0} \int_M \Sigma_\alpha u dv_g = \int_M u dv_g.$$

Multiplying (3.1) by u, integrating over M, and passing to the limit as $\alpha \to \alpha_0$, we then get that

$$\int_M |\nabla u|^2 dv_g + \alpha_0 \left(\int_M u dv_g\right)^2 = \lambda \int_M u^{2^\star} dv_g.$$

Hence,

$$\frac{1}{K_n^2} \le \frac{\int_M |\nabla u|^2 dv_g + \alpha_0 \left(\int_M u dv_g\right)^2}{\left(\int_M u^{2^\star} dv_g\right)^{2/2^\star}} \le \lambda \left(\int_M u^{2^\star} dv_g\right)^{1 - \frac{2}{2^\star}}. \tag{3.4}$$

Since $\lambda \le K_n^{-2}$ and

$$\int_M u^{2^\star} dv_g \le 1 = \liminf_{\alpha \to \alpha_0} \int_M u_\alpha^{2^\star} dv_g$$

it follows from (3.4) that $\lambda = K_n^{-2}$ and $\|u\|_{2^\star} = 1$. In particular, u is an extremal function for (1.4), and the above claim is proved. Summarizing, the proof of Theorem 1 reduces to the proof that (3.3) is impossible. We proceed by contradiction, and assume that (3.3) holds. Then the estimates 1 to 5 of Section 2 hold also. For $\delta > 0$ small, and $x \in B_0(\delta)$, we set

$$g_\alpha(x) = \exp_{x_\alpha}^\star g(x) \quad \text{and} \quad v_\alpha(x) = u_\alpha\left(\exp_{x_\alpha}(x)\right).$$

We also let η be a smooth cut-off function such that $\eta = 1$ in $B_0(\delta/2)$, $\eta = 0$ in $\mathbb{R}^n \backslash B_0(\delta)$, $|\nabla\eta| \le C\delta^{-1}$, and $|\nabla^2\eta| \le C\delta^{-2}$, where, as in what follows, $C > 0$ is a constant independent of α and δ. By the definition of K_n,

$$\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star}\, dx\right)^{\frac{2}{2^\star}} \le K_n^2 \int_{B_0(\delta)} |\nabla(\eta v_\alpha)|^2 dx. \tag{3.5}$$

We have that

$$\int_{B_0(\delta)} |\nabla(\eta v_\alpha)|^2 dx \le \int_{B_0(\delta)} \eta^2 v_\alpha \Delta v_\alpha dx + C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx$$

and

$$\Delta v_\alpha = \Delta_{g_\alpha} v_\alpha + (g_\alpha^{ij} - \delta^{ij})\partial_{ij} v_\alpha - g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k \partial_k v_\alpha$$

where Δ is the Euclidean Laplacian, δ^{ij} is the Kronecker symbol, and the $\Gamma(g_\alpha)_{ij}^k$'s are the Christoffel symbols of the Levi-Civita connection with respect to g_α. Hence,

$$\begin{aligned}\int_{B_0(\delta)} |\nabla(\eta v_\alpha)|^2 dx &\le \int_{B_0(\delta)} \eta^2 v_\alpha \Delta_{g_\alpha} v_\alpha dx + C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx \\ &+ \int_{B_0(\delta)} \eta^2 v_\alpha (g_\alpha^{ij} - \delta^{ij})\partial_{ij} v_\alpha dx - \int_{B_0(\delta)} \eta^2 v_\alpha g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k \partial_k v_\alpha dx.\end{aligned}$$

Integrating by parts, and thanks to (3.1), we then get that

$$\begin{aligned}\int_{B_0(\delta)} |\nabla(\eta v_\alpha)|^2 dx &\le \frac{1}{K_n^2}\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \alpha\int_M u_\alpha dv_g \int_{B_0(\delta)} \eta^2 v_\alpha dx \\ &+ C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx - \int_{B_0(\delta)} \eta^2 (g_\alpha^{ij} - \delta^{ij})\partial_i v_\alpha \partial_j v_\alpha dx \\ &+ \frac{1}{2}\int_{B_0(\delta)} \left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k) + \partial_{ij} g_\alpha^{ij}\right)\eta^2 v_\alpha^2 dx.\end{aligned}$$

By (3.5), this implies in particular that

$$\begin{aligned}0 &\le \int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{\frac{2}{2^\star}} \\ &+ \frac{1}{2}K_n^2 \int_{B_0(\delta)} \left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k) + \partial_{ij} g_\alpha^{ij}\right)\eta^2 v_\alpha^2 dx \\ &- K_n^2 \int_{B_0(\delta)} \eta^2 (g_\alpha^{ij} - \delta^{ij})\partial_i v_\alpha \partial_j v_\alpha dx + C\delta^{-2}\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx.\end{aligned} \tag{3.6}$$

Now, we may proceed as in Djadli and Druet [4]. By (2.24) and (2.28),

$$\lim_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx}{\int_{B_0(\delta)} v_\alpha^2 dx} = 0. \tag{3.7}$$

Similarly, since $x_\alpha \to x_0$ as $\alpha \to \alpha_0$, the Cartan expansion of g gives that

$$\lim_{\alpha\to\alpha_0} \left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k) + \partial_{ij} g_\alpha^{ij}\right)(0) = \frac{1}{3}S_g(x_0)$$

where S_g is the scalar curvature of g. By (2.24) and (2.28) we then get that

$$\limsup_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)} \left(\partial_k(g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k) + \partial_{ij} g_\alpha^{ij}\right)\eta^2 v_\alpha^2 dx}{\int_{B_0(\delta)} v_\alpha^2 dx} = \frac{1}{3}S_g(x_0) + \varepsilon_\delta \tag{3.8}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. Independently, we claim that when $S_g(x_0) \le 0$,

$$\limsup_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{\frac{2}{2^\star}}}{\int_{B_0(\delta)} v_\alpha^2 dx} \le \varepsilon_\delta \tag{3.9}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. By Hölder's inequalities we indeed do have that

$$\begin{aligned}&\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{\frac{2}{2^\star}} \\ &\quad\le \left(\left(\int_{B_0(\delta)} v_\alpha^{2^\star} dx\right)^{(2^\star-2)/2^\star} - 1\right)\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{2/2^\star}\end{aligned}$$

and thanks to the Cartan expansion of g,

$$dx \le \left(1 + \frac{1}{6}R_{ij}(x_\alpha)x^i x^j + C|x|^3\right) dv_{g_\alpha}$$

where the $R_{ij}(x_\alpha)$'s are the components of the Ricci curvature at x_α in the exponential chart. It follows from these equations that

$$\begin{aligned}&\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star} dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{\frac{2}{2^\star}} \\ &\quad\le \left(\frac{1+\varepsilon_\delta}{3n} X_\alpha + \varepsilon_\delta \int_{B_0(\delta)} |x|^2 v_\alpha^{2^\star} dv_{g_\alpha}\right)\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx\right)^{2/2^\star}\end{aligned}$$

where

$$X_\alpha = R_{ij}(x_\alpha) \int_{B_0(\delta)} x^i x^j v_\alpha^{2^\star} dv_{g_\alpha}.$$

By (2.14) we have that

$$\int_{B_0(\delta)} |x|^2 v_\alpha^{2^\star} dv_{g_\alpha} \le C \int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}$$

and by (2.23) we have that for any $R > 0$,

$$R_{ij}(x_\alpha) \int_{B_0(\delta)\setminus B_0(R\mu_\alpha)} x^i x^j v_\alpha^{2^\star} dv_{g_\alpha} \le \varepsilon_R \int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$. Noting that

$$\frac{R_{ij}(x_\alpha)\int_{B_0(R\mu_\alpha)} x^i x^j v_\alpha^{2^\star} dv_{g_\alpha}}{\int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}} = \frac{R_{ij}(x_\alpha)\int_{B_0(R)} x^i x^j \tilde{u}_\alpha^{2^\star} dv_{\tilde{g}_\alpha}}{\int_{B_0(\delta\mu_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{g_\alpha}}$$

where $\tilde{u}_\alpha$ is as in the proof of Estimate 1, and that

$$\int_{B_0(R)} x^i x^j \tilde{u}^{2^\star} dx = \frac{1}{n}\delta^{ij} \int_{B_0(R)} |x|^2 \tilde{u}^{2^\star} dx$$

where $\tilde{u}$ is as in (2.13) and the δ^{ij}'s are the Kroenecker symbols, we get that

$$\limsup_{\alpha\to\alpha_0} \frac{\frac{1+\varepsilon_\delta}{3n} X_\alpha + \varepsilon_\delta \int_{B_0(\delta)} |x|^2 v_\alpha^{2^\star} dv_{g_\alpha}}{\int_{B_0(\delta)} v_\alpha^2 dv_{g_\alpha}} \le \varepsilon_\delta$$

when $S_g(x_0) \le 0$. By (2.6) we also have that

$$\lim_{\alpha\to\alpha_0} \int_{B_0(\delta)} (\eta v_\alpha)^{2^\star} dx = 1$$

and, combining these equations, we get (3.9). At last (we refer to Djadli and Druet [4] for details) it can be proved with (2.23), (2.24) and (2.28) that for any $R > 0$,

$$\limsup_{\alpha\to\alpha_0} \frac{\left|\int_{B_0(\delta)} \eta^2 (g_\alpha^{ij} - \delta^{ij}) \partial_i v_\alpha \partial_j v_\alpha dx\right|}{\int_{B_0(\delta)} v_\alpha^2 dx}$$
$$\le \varepsilon_R + \varepsilon_\delta + \frac{C}{R^{n-4}} \limsup_{\alpha\to\alpha_0} \frac{1}{\int_{B_0(\delta\mu_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha}}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$, $\varepsilon_\delta \to 0$ as $\delta \to 0$, and $\tilde{u}_\alpha$ is as in the proof of estimate 1. Noting that by (2.11) and (2.12),

$$\liminf_{\alpha\to\alpha_0} \int_{B_0(\delta\mu_\alpha^{-1})} \tilde{u}_\alpha^2 dv_{\tilde{g}_\alpha} \ge \int_{B_0(\tilde{R})} \tilde{u}^2 dx$$

for any $\tilde{R} > 0$, where $\tilde{u}$ is as in (2.13), and that if $n = 4$,

$$\lim_{\tilde{R}\to+\infty} \int_{B_0(\tilde{R})} \tilde{u}^2 dx = +\infty$$

it follows that

$$\limsup_{\alpha\to\alpha_0} \frac{\int_{B_0(\delta)} \eta^2 (g_\alpha^{ij} - \delta^{ij}) \partial_i v_\alpha \partial_j v_\alpha dx}{\int_{B_0(\delta)} v_\alpha^2 dx} = \varepsilon_\delta \tag{3.10}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. Combining (3.6)-(3.10) we then get that

$$\frac{1}{6} K_n^2 S_g(x_0) + \varepsilon_\delta \ge 0 \tag{3.11}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. Since $S_g(x_0) < 0$, (3.11) is impossible, and the contradiction follows. As already mentioned, this proves Theorem 1.

4. Proof of Theorem 2

We start with the proof that (1.3) is valid if $S_g \le 0$ and g is conformally flat. Let α be a sequence of positive real numbers converging to $+\infty$. We assume by contradiction that for any α in the sequence,

$$\inf_{H_1^2(M)\backslash\{0\}} I_\alpha(u) < \frac{1}{K_n^2} \tag{4.1}$$

where I_α is as in Section 1. Thanks to Section 1, we get that there exist $u_\alpha \in H_1^2(M)$ and $\Sigma_\alpha \in L^\infty(M)$, $0 \le \Sigma_\alpha \le 1$, such that

$$\Delta_g u_\alpha + \alpha \left(\int_M u_\alpha dv_g \right) \Sigma_\alpha = \lambda_\alpha u_\alpha^{2^\star - 1} \tag{4.2}$$

and $\int_M u_\alpha^{2^\star} dv_g = 1$. Moreover, $\Sigma_\alpha \varphi = \varphi$ for any $\varphi \in H_1^2(M)$ such that $|\varphi| \le C u_\alpha$ for some constant $C > 0$. As already mentioned, we necessarily have that

$$\lim_{\alpha \to +\infty} \int_M u_\alpha^2 dv_g = 0 \tag{4.3}$$

and Estimates 1 to 5 of Section 2 hold. In addition to these estimates, we claim that L^1-concentration holds also for the u_α's.
ESTIMATE 6. For any $\delta > 0$,

$$\lim_{\alpha \to +\infty} \frac{\int_{M \backslash B_{x_0}(\delta)} u_\alpha dv_g}{\int_M u_\alpha dv_g} = 0 \tag{4.4}$$

where x_0 is the concentration point of the sequence (u_α).

Proof of Estimate 6. We let $0 \le \eta \le 1$ be a smooth function such that $\eta = 1$ in $M \backslash B_{x_0}(\delta)$ and $\eta = 0$ in $B_{x_0}(\delta/2)$. We have that

$$\int_M u_\alpha dv_g \int_{M \backslash B_{x_0}(\delta)} u_\alpha dv_g \le \int_M u_\alpha dv_g \int_M \eta u_\alpha dv_g \tag{4.5}$$

and as when proving estimate 4, we get by the Moser iterative scheme that

$$\begin{aligned} \int_M u_\alpha dv_g \int_M \eta u_\alpha dv_g &= \frac{1}{\alpha} \int_M \eta u_\alpha \left(\lambda_\alpha u_\alpha^{2^\star - 1} - \Delta_g u_\alpha \right) dv_g \\ &\le \frac{1}{\alpha} \lambda_\alpha \int_M \eta u_\alpha^{2^\star} dv_g + \frac{1}{\alpha} \int_M |(\nabla \eta \nabla u_\alpha)| u_\alpha dv_g \\ &\le \varepsilon_\alpha \left(\int_M u_\alpha dv_g \right)^2 + \frac{C}{\alpha} X_\alpha \int_M u_\alpha dv_g \end{aligned} \tag{4.6}$$

where $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, $C > 0$ is independent of α, and

$$X_\alpha = \sqrt{\int_{M \backslash B_{x_0}(\delta/2)} |\nabla u_\alpha|^2 dv_g}.$$

The proof of (4.4) then reduces to the proof that for $\hat{\delta} > 0$ small, there exists $C > 0$, independent of α, such that

$$\int_{M \backslash B_{x_0}(\hat{\delta})} |\nabla u_\alpha|^2 dv_g \le C \left(\int_M u_\alpha dv_g \right)^2. \tag{4.7}$$

As above, let $0 \le \eta \le 1$ be a smooth function such that $\eta = 1$ in $M \backslash B_{x_0}(\hat{\delta})$ and $\eta = 0$ in $B_{x_0}(\hat{\delta}/2)$. Multiplying (4.2) by $\eta^2 u_\alpha$ and integrating over M, we get that

$$\int_M \eta^2 |\nabla u_\alpha|^2 dv_g + 2 \int_M \eta u_\alpha (\nabla \eta \nabla u_\alpha) dv_g \le \lambda_\alpha \int_M \eta^2 u_\alpha^{2^\star} dv_g.$$

Therefore, by the Moser iterative scheme,

$$\begin{aligned}\int_M \eta^2|\nabla u_\alpha|^2 dv_g &\le C\int_M |\eta\nabla u_\alpha|u_\alpha dv_g + \lambda_\alpha\int_M \eta^2 u_\alpha^{2^\star} dv_g\\ &\le C\int_M u_\alpha dv_g\sqrt{\int_M \eta^2|\nabla u_\alpha|^2 dv_g} + \lambda_\alpha\int_M \eta^2 u_\alpha^{2^\star} dv_g\end{aligned}$$

and we get that

$$\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \le C\sqrt{\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2}} + \lambda_\alpha\frac{\int_M \eta^2 u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2}.$$

By the Moser iterative scheme that we apply once again,

$$\lim_{\alpha\to+\infty}\frac{\int_M \eta u_\alpha^{2^\star} dv_g}{\left(\int_M u_\alpha dv_g\right)^2} = 0$$

and it follows that

$$\limsup_{\alpha\to+\infty}\frac{\int_M \eta^2|\nabla u_\alpha|^2 dv_g}{\left(\int_M u_\alpha dv_g\right)^2} \le C^2.$$

In particular, (4.7) holds, and we get from (4.5) and (4.6) that (4.4) holds also. This proves L^1-concentration for the u_α's, and Estimate 6. □

We proceed now with the proof of the first part of Theorem 2. Since (M,g) is conformally flat, there exists $\varphi\in C^\infty(M)$, $\varphi>0$, such that $\tilde g = \varphi^{4/(n-2)}g$ is flat around x_0. Set $v_\alpha = \varphi^{-1}u_\alpha$. By conformal invariance of the conformal Laplacian,

$$\Delta_{\tilde g} v_\alpha + \left(\alpha\int_M u_\alpha dv_g\Sigma_\alpha - \frac{n-2}{4(n-1)}S_g u_\alpha\right)\varphi^{-\frac{n+2}{n-2}} = \lambda_\alpha v_\alpha^{2^\star-1}. \tag{4.8}$$

We let $\delta>0$ be such that $\tilde g$ is flat in $B_{x_0}(2\delta)$, the ball with respect to $\tilde g$ of center x_0 and radius 2δ, and we assimilate $B_{x_0}(2\delta)$ and g with $B_0(2\delta)$ and ξ, where ξ is the Euclidean metric. We also let $0\le\eta\le 1$ be a smooth cut-off function such that $\eta=1$ in $B_0(\delta/2)$ and $\eta=1$ in $\mathbb{R}^n\backslash B_0(\delta)$. By the defintion of K_n,

$$\left(\int_{B_0(\delta)}(\eta v_\alpha)^{2^\star}dx\right)^{\frac{2}{2^\star}} \le K_n^2\int_{B_0(\delta)}|\nabla(\eta v_\alpha)|^2 dx \tag{4.9}$$

and we have that

$$\int_{B_0(\delta)}|\nabla(\eta v_\alpha)|^2dx \le \int_{B_0(\delta)}\eta^2 v_\alpha\Delta v_\alpha dx + C\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx$$

where $C>0$ is independent of α. Hence, by (4.8), (4.9), and since $S_g\le 0$,

$$\begin{aligned}&\left(\int_{B_0(\delta)}(\eta v_\alpha)^{2^\star}dx\right)^{\frac{2}{2^\star}} + \alpha K_n^2\int_M u_\alpha dv_g\int_{B_0(\delta)}\eta^2\varphi^{-2^\star}u_\alpha dx\\ &\qquad\le \lambda_\alpha K_n^2\int_{B_0(\delta)}\eta^2 v_\alpha^{2^\star}dx + C\int_{B_0(\delta)\backslash B_0(\delta/2)} v_\alpha^2 dx.\end{aligned}\tag{4.10}$$

On the one hand, $\lambda_\alpha K_n^2 \le 1$. On the other hand, it follows from Hölder's inequalities that

$$\int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star}\, dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star}\, dx\right)^{\frac{2}{2^\star}}$$
$$\le \left(\left(\int_{B_0(\delta)} v_\alpha^{2^\star}\, dx\right)^{1-\frac{2}{2^\star}} - 1\right)\left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star}\, dx\right)^{\frac{2}{2^\star}}.$$

Moreover,

$$\int_{B_0(\delta)} v_\alpha^{2^\star}\, dx = \int_{B_{x_0}(\delta)} u_\alpha^{2^\star} \varphi^{-2^\star}\, dv_{\tilde g} \le \int_M u_\alpha^{2^\star}\, dv_g = 1$$

and it follows that

$$\lambda_\alpha K_n^2 \int_{B_0(\delta)} \eta^2 v_\alpha^{2^\star}\, dx - \left(\int_{B_0(\delta)} (\eta v_\alpha)^{2^\star}\, dx\right)^{\frac{2}{2^\star}} \le 0.$$

Coming back to (4.10), we then get that

$$\alpha K_n^2 \int_M u_\alpha dv_g \int_{B_0(\delta)} \eta^2 \varphi^{-2^\star} u_\alpha dx \le C \int_{B_0(\delta)\setminus B_0(\delta/2)} v_\alpha^2 dx. \tag{4.11}$$

We have that

$$\int_{B_0(\delta)} \eta^2 \varphi^{-2^\star} u_\alpha dx \ge \int_{B_{x_0}(\hat\delta)} u_\alpha dv_g$$

for some $\hat\delta > 0$ small, and by the Moser iterative scheme, there exists $C > 0$, independent of α, such that for $\hat\delta > 0$ sufficiently small,

$$\int_{B_0(\delta)\setminus B_0(\delta/2)} v_\alpha^2 dx \le C \int_M u_\alpha dv_g \int_{M\setminus B_{x_0}(\hat\delta)} u_\alpha dv_g.$$

By (4.11) we then get that

$$\alpha K_n^2 \le C \frac{\int_{M\setminus B_{x_0}(\hat\delta)} u_\alpha dv_g}{\int_{B_{x_0}(\hat\delta)} u_\alpha dv_g} \tag{4.12}$$

while by (4.4), the right-hand side of (4.12) goes to 0 as $\alpha \to +\infty$. A contradiction, so that (1.3) is true if (M, g) is conformally flat and the scalar curvature of g is nonpositive. The first part of Theorem 2 is proved.

We prove now the second part of Theorem 2, namely that (1.3) is not valid if $n \ge 6$ and g is scalar flat in an open neighbourhood of one nonconformally flat point. We let W_g be the Weyl tensor of g, and Rc_g be the Ricci curvature of g. By assumption, there exists $x \in M$ such that $W_g(x) \not\equiv 0$ and $S_g \equiv 0$ in $B_x(\delta_0)$ for some $\delta_0 > 0$. We let $\tilde g$ be a conformal metric to g such that $Rc_{\tilde g}(x) \equiv 0$. We also let $\delta > 0$ be such that $B_x(\delta)$ with respect to $\tilde g$ is a subset of $B_x(\delta_0)$ with respect to g. Since the Weyl curvature tensor is a conformal invariant, $W_{\tilde g}(x) \not\equiv 0$. Given

$B > 0$, it follows from the conformal invariance of the conformal Laplacian that

$$\inf_{u \in H_1^2(M)\setminus\{0\}} \frac{\int_M |\nabla u|^2 dv_g + B\left(\int_M |u| dv_g\right)^2}{\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star}}$$
$$\leq \inf_{u \in \mathcal{H}} \frac{\int_M |\nabla u|^2 dv_{\tilde{g}} + \frac{n-2}{4(n-1)} \int_M S_{\tilde{g}} u^2 dv_{\tilde{g}} + \hat{B}\left(\int_M |u| dv_{\tilde{g}}\right)^2}{\left(\int_M |u|^{2^\star} dv_{\tilde{g}}\right)^{2/2^\star}}$$

where $\hat{B} > 0$, and $\mathcal{H}$ consists of the nonzero functions $u \in H_1^2(M)$ which are such that $\mathrm{Supp} u \subset B_x(\delta)$. For $\varepsilon > 0$, we define

$$u_\varepsilon = \left(\varepsilon + r^2\right)^{1-\frac{n}{2}} - \left(\varepsilon + \delta^2\right)^{1-\frac{n}{2}} \quad \text{if } r \leq \delta,$$
$$u_\varepsilon = 0 \ \text{ otherwise},$$

where $r = d_{\tilde{g}}(x, .)$. Then,

$$\int_M u_\varepsilon dv_{\tilde{g}} \leq C$$

where $C > 0$ is independent of ε, and it follows from the computations in Aubin [1] that for any $\hat{B} > 0$,

$$\frac{\int_M |\nabla u_\varepsilon|^2 dv_{\tilde{g}} + \frac{n-2}{4(n-1)} \int_M S_{\tilde{g}} u_\varepsilon^2 dv_{\tilde{g}} + \hat{B}\left(\int_M |u_\varepsilon| dv_{\tilde{g}}\right)^2}{\left(\int_M |u_\varepsilon|^{2^\star} dv_{\tilde{g}}\right)^{2/2^\star}}$$
$$\leq \frac{1}{K_n^2}\left(1 - C_1 |W_{\tilde{g}}(x)|^2 \varepsilon^2 + o\left(\varepsilon^2\right)\right) \quad \text{if } n > 6$$
$$\leq \frac{1}{K_6^2}\left(1 + C_2 |W_{\tilde{g}}(x)|^2 \varepsilon^2 \ln \varepsilon + o\left(\varepsilon^2 \ln \varepsilon\right)\right) \quad \text{if } n = 6$$

where C_1 and C_2 are explicit positive constants which do not depend on ε. Hence, for any $B > 0$,

$$\inf_{u \in H_1^2(M)\setminus\{0\}} \frac{\int_M |\nabla u|^2 dv_g + B\left(\int_M |u| dv_g\right)^2}{\left(\int_M |u|^{2^\star} dv_g\right)^{2/2^\star}} < \frac{1}{K_n^2}$$

and this proves that if $n \geq 6$ and g is scalar flat in an open neighbourhood of one nonconformally flat point, then inequality (1.3) is not valid. This ends the proof of Theorem 2.

Acknowledgements

The author expresses his gratitude to Olivier Druet and Michel Vaugon for stimulating discussions on this work.

References

[1] Aubin, T., Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.*, 55, 269-296, 1976. MR **55:**4288

[2] Caffarelli, L.A., Gidas, B. and Spruck, J., Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Communications on Pure and Applied Mathematics, 42, 271-297, 1989. MR **90c:**35075

[3] Croke, C.B., A sharp four dimensional isoperimetric inequality, *Comment. Math. Helv.*, 59, 1984, 187-192. MR **85f:**53060

[4] Djadli, Z. and Druet, O., Extremal functions and Sobolev inequalities on compact manifolds, *Calc. Var. Partial Differential Equations*, 12, 59–84, 2001. CMP 2001:07

[5] Druet, O., The best constants problem in Sobolev inequalities, *Math. Ann.*, 314, 327-346, 1999. MR **2000d:**58033

[6] Druet, O., and Hebey, E., The *AB* program in geometric analysis. Sharp Sobolev inequalities and related problems, *Preprint*, 2000.

[7] Druet, O., Hebey, E., and Vaugon, M., Sharp Sobolev inequalities with lower order remainder terms, *Trans. Amer. Math. Soc.*, 353, 269-289, 2001. MR **2001g:**58060

[8] Druet, O., and Robert, F., Asymptotic profile and blow-up estimates on compact Riemannian manifolds, *Preprint*, 1999.

[9] Gilbarg, D. and Trudinger, N.S., *Elliptic partial differential equations of second order*, Grundlehren der Mathematischen Wissenschaften, 224, Springer-Verlag, Second Edition, 1983. MR **86C:**35035

[10] Hebey, E., *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, CIMS Lecture Notes, Courant Institute of Mathematical Sciences, Volume 5, 1999. MR **2000e:**58011

[11] Hebey, E. and Vaugon, M., The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds, *Duke Math. J.*, 79, 1995, 235-279. MR **96c:**53057

[12] Kleiner, B., An isoperimetric comparison theorem, *Invent. Math.*, 108, 1992, 37-47. MR **92M:**53056

[13] Obata, M., The conjectures on conformal transformations of Riemannian manifolds, *J. Differential Geom.*, 6, 247-258, 1971. MR **46:**2601

[14] Weil, A., Sur les surfaces à courbure négative, *C. R. Acad. Sci. Paris*, 182, 1926, 1069-1071.

Département de Mathématiques, Site de Saint-Martin, Université de Cergy-Pontoise, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France

E-mail address: `Emmanuel.Hebey@math.u-cergy.fr`